if $y_e \geq 1/\varkappa$. Note that (34) determines the transient time in a column closed at both ends and, consequently, by an appropriate choice of the rate of circulation ($\varkappa$) of the apparatus operating in the scheme illustrated in Fig. 1c, the same equilibrium concentration can be achieved in a much shorter time.

NOTATION

σ, mass flow rate of the liquid through the channel in unit time; D, binary diffusion coefficient; α, thermal diffusion constant; ρ, density; β, volume expansion coefficients; δ, gap, i.e., the distance between the two constant-temperature surfaces; $\Delta T = T_2 - T_1$; T_2, T_1, temperatures of the heated and cooled surfaces $T = (T_1 + T_2)/2$; η, dynamic viscosity; B, length; L, height of the apparatus; z, vertical coordinate; x, longitudinal coordinate; $K = g^2\rho^3\beta^2\delta^7(\Delta T)^2 B/9!\eta^2 D$; c, mass concentration; τ^*, transfer of the purposeful component in the vertical direction in units of mass in unit time per unit length of the apparatus; $m^* = \rho\delta$; t, time; f, area of transverse cross section of the channel; w, rate of flow of the liquid in the channel; $y_e = HL/K$; ω, θ, ξ, $\varkappa$, see Eqs. (8); u, see Eq. (12); and p, an operator. Subscripts: e, upper channel; i, lower channel; 0, initial value, and K, end of the apparatus.

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STABILITY OF THE INTERPHASE SURFACE IN THE FREEZING OF MOIST GROUND

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It is suggested that the formation of ice layers should be regarded as a consequence of a loss of stability of the motion of the freezing front. The kinetics of the freezing process is investigated and a stability criterion is obtained.

The freezing of moist ground is accompanied by the migration of moisture, i.e., by a redistribution of the initial moisture. Experiments show that in different grounds and for different modes of freezing the redistribution of the moisture leads to various textures of the frozen rocks. In particular, in certain cases ice layers are formed, and sometimes monotonic freezing, etc., occurs. There are different ways of describing the various aspects of this phenomenon. Examples are given in [1, 2]. Below we carry out a theoretical analysis based on a study of the stability of the process. The formation of ice layers is treated as a consequence of the loss of stability of the motion of the freezing front with respect to small perturbations.

The one-dimensional freezing problem, taking into account the migration of moisture, can be described in the following form [3]:

$$\frac{\partial T_1}{\partial t} = a_1 \frac{\partial^2 T_1}{\partial x^2}, \quad 0 < x < s(t),$$

$$\left.\begin{aligned} \frac{\partial T_2}{\partial t} &= a_2 \frac{\partial^2 T_2}{\partial x^2} \\ \frac{\partial W}{\partial t} &= K \frac{\partial^2 W}{\partial x^2} \end{aligned}\right\} \quad s(t) < x < L.$$

The initial conditions are

$$W(x, 0) = W_0, \quad T_2(x, 0) = T_н,$$

and the boundary conditions are

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$$\text{for } x=0 \quad \lambda_1 \frac{\partial T_1}{\partial x} = Q_1,$$

$$x=L \quad \lambda_2 \frac{\partial T_2}{\partial x} = Q_2, \; W(L,\, t) = W_0,$$

$$\text{for } \; x = s(t)$$

$$T_1(s(t),\, t) = T_2(s(t),\, t) = T_0, \; q = K\frac{\partial W}{\partial x},$$

$$\lambda_1 \frac{\partial T_1}{\partial x} - \lambda_2 \frac{\partial T_2}{\partial x} - \varkappa q = \varkappa v (W - W_{\text{H}}).$$

To a first approximation we can assume that the thermal parameters are independent of the temperature and the moisture content. In addition, we will use the quasi-stationary hypothesis for the temperature field. Then, the equations for the temperature distribution can be integrated and the problem can be formulated as follows:

$$\frac{\partial W}{\partial t} = K \frac{\partial^2 W}{\partial x^2},$$

$$K \frac{\partial W}{\partial x}\bigg|_{x=s(t)} = q, \; W(L,\, t) = W_0, \; W(x,\, 0) = W_0.$$

On the movable boundary the Stefan condition is specified as follows taking migration into account:

$$A - \varkappa q = \varkappa v (W - W_{\text{H}}).$$

Here $A = \lambda_1(\partial T_1/\partial x) - \lambda_2(\partial T_2/\partial x) = \text{const}$ in view of the quasi-stationary hypothesis.

There is an expression for q in the literature of the form $q = \alpha(W - W_H)$. This condition determines the flow from the melted zone into the frozen zone. However, one must take into account that for the same difference in the moisture content the migration flow depends on the amount of ice in the frozen front. The presence of ice on the left of the front leads to an additional inflow of liquid into the frozen zone. This is pointed out in [4, p. 65] in an experimental study of the migration of moisture into mountain rocks. A similar phenomenon was observed in [5]. The additional inflow of liquid will be taken into account here in the following form:

$$q = K\frac{\partial W}{\partial x} = (\alpha + f(l))(W - W_{\text{H}}),$$

where $f(l)$ is a monotonically increasing function of the ice content, and $l = W - W_H + q/v$ is the ice content of the frozen front. From Stefan's condition on the movable boundary we can write

$$l = \frac{A}{\varkappa v}. \tag{1}$$

It is of interest to analyze the stability of the stationary solution with respect to two types of perturbations. The first case corresponds to the application of a perturbation to the moisture content in the case of a plane frozen front. The loss in stability here may lead to the fromation of icy streaks parallel to the front. The second case represents the stability of the solution with respect to a small deformation of the shape of the frozen boundary and represents the possibility of "icy tongues" forming, i.e., layers of ice perpendicular to the front. We investigated the general case when the moisture content varies and the shape of the frozen front is simultaneously deformed. This enables a single criterion to be obtained reflecting the stability of the solution with respect to the two different forms of perturbation.

In the light of the above we will formulate the problem which describes the one-dimensional migration of moisture into the melted zone when the frozen front is displaced and when A is the difference in the heat fluxes at the point, and $x = s(t)$ is a constant

$$\frac{\partial W}{\partial t} = K\left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2}\right) \; s(t) < x < \infty, \; -\infty < y < \infty, \tag{2}$$

$$W(x,\, y,\, 0) = W_0,$$

$$\lim_{x\to\infty} W = W_0, \quad \lim_{y\to\pm\infty} W < \infty,$$

$$A - \varkappa q = \varkappa v (W - W_{н}) \quad (3)$$

$$\left.\begin{aligned} & A - \varkappa q = \varkappa v (W - W_{н}) \\ & K \frac{\partial W}{\partial x} = q = (\alpha + F(v))(W - W_{н}) \end{aligned}\right\} x = s(t), \quad (4)$$

where F(v) is a monotonically decreasing function, and $F(v) = f(A\varkappa/v)$. The mode of freezing in which the moisture content and the ice content in the front are constant, will be called stationary. The stationary solution $W(x-v_1 t, y)$ will be written in the form

$$\frac{W - W_0}{W_{к} - W_0} = \exp\left(-\frac{v_1 (x - v_1 t)}{K}\right), \quad (5)$$

where W_K is the stationary moisture content on the front, and v_1 is the stationary velocity directed along the x axis. This solution satisfied the heat-conduction equation (2).

Below we will investigate the stability of the solution (5) by the small-perturbation method. The method of obtaining the main relations in this method is taken from [6].

The perturbed distribution of the moisture content has the following form:

$$W = W_1 + \varepsilon \cdot \exp(mx + ipy + nt), \quad (6)$$

where W_1 is the unperturbed distribution of the moisture content, m and n are complex constants, p is a real constant, and ε is a small quantity.

The perturbation in Eq. (6) is periodic along the y coordinate with a frequency p, and disappears as $x \to \infty$, i.e., the perturbed distribution of the moisture content satisfies the following condition on the right boundary:

$$\lim_{x \to \infty} W = W_0.$$

This is always satisfied when m satisfies the inequality $\operatorname{Re} m < 0$. Otherwise the values of m in this paper will be called extraneous. Substituting the right side of Eq. (6) into Eq. (2) we obtain $n = K(m^2 - p^2)$. The moisture content, the migration flow on the movable boundary, and the velocity of the front can be represented in the usual perturbed form

$$W = W_{к} + W_2 \exp(\Omega t + ipy), \quad (7)$$

$$q = q_1 + q_2 \exp(\Omega t + ipy), \quad (8)$$

$$v = v_1 + v_2 \exp(\Omega t + ipy). \quad (9)$$

Here W_K, q_1, and v_1 are the unperturbed values of the moisture content and the migration flow on the frozen front and the velocity of the front, and in addition,

$$\Omega = m v_1 + K(m^2 - p^2). \quad (10)$$

Substituting (6)-(9) into Eq. (2) as well as the conditions (3) and (4) we obtain the following relations:

$$v_2 = -\frac{K\Omega}{q_1}(\varepsilon - W_2),$$

$$q_2 = K\left(m\varepsilon + (\varepsilon - W_2)\frac{v_1}{K}\right),$$

$$q_2 + (W_{к} - W_{н}) v_2 + v_1 W_2 = 0,$$

$$q_2 = (W_{к} - W_{н}) F_1' v_2 + (\alpha + F_1) W_2.$$

The value of m is found from the condition for the last four uniform linear equations with the four unknown q_2, v_2, W_2, and ε to be compatible, namely,

$$\begin{vmatrix} 0 & 1 & -\frac{K\Omega}{q_1} & \frac{K\Omega}{q_1} \\ 1 & 0 & v_1 & -K\left(m + \frac{v_1}{K}\right) \\ 1 & -F_1'(W_{к} - W_{н}) & -(\alpha + F_1) & 0 \\ 1 & W_{к} - W_{н} & v_1 & 0 \end{vmatrix} = 0.$$

Expanding this determinant we obtain the equation

$$\Omega = \frac{am+b}{cm+d} \tag{11}$$

or, taking (10) into account

$$a_0 m^3 + a_1 m^2 + a_2 m + a_3 = 0, \tag{12}$$

where $a_0 = Kc$; $a_1 = cv_1 + Kd$; $a_2 = dv_1 - Kcp^2 - a$; $a_3 = Kp^2 d - b$; $a = -F_1(F_1 + v_2)$; $b = -(v_1/K)F_1(F_1 + v_1)$, $c = K(1 + F'_1)$; $d = -(F_1 - v_1 F'_1)$. Separating the real and imaginary parts in (10) we obtain the following relations:

$$\operatorname{Re}\Omega = (v_1 + K \operatorname{Re} m)\operatorname{Re} m - K(p^2 + \operatorname{Im}^2 m), \tag{13}$$

$$\operatorname{Im}\Omega = \operatorname{Im} m\,(v_1 + 2K \operatorname{Re} m). \tag{14}$$

Bearing in mind that easily verified inequality $ad - bc > 0$, in view of the well-known property [7] of fractional-linear transformations of the form (11), we can write

$$\operatorname{sign}(\operatorname{Im}\Omega) = \operatorname{sign}(\operatorname{Im} m). \tag{15}$$

Comparing (14) and (15) we can write for the roots of system (10) and (11), having nonzero imaginary parts, the inequality

$$\operatorname{Re} m > -\frac{v_1}{2K}.$$

At the same time, for any m which satisfies the relations

$$-\frac{v_1}{K} \leqslant \operatorname{Re} m < 0,$$

using (13) we can obtain

$$\operatorname{Re}\Omega \leqslant -K(\operatorname{I}^2\mathrm{m}\, m + p^2) < 0.$$

Consequently, the complex roots m correspond to the case $\operatorname{Re}\Omega < 0$, or, which is the same thing, unstable solutions of the initial problem correspond to real roots of Eq. (12).

Owing to the continuous dependence of the roots m_i on the initial parameters, the transition from a stable solution to an unstable solution ($c \operatorname{Re} m < 0$) may occur either for m real and $\Omega = 0$, or in the case of a change in the sign of the coefficient a_0 in (12). In the latter case one of the roots changes from $-\infty$ to $+\infty$ or vice versa. It is easy to show that system (10), (11) has no real solutions m for $\Omega = 0$. Consequently, this transition is only possible when the sign of a_0 changes.

We will consider the degeneration of Eq. (12) as $a_0 \to 0$, i.e., when $F'_1 = -1$, separately. One of the roots m_1 in this case approaches $+\infty$ or $-\infty$ depending on the sign of a_0/a_1 (see, e.g., [8]). If $F'_1 \to -1+0$, this root goes to $+\infty$, as was mentioned above, or remains stable or extraneous (i.e., $\operatorname{Re} m > 0$) over the whole region $1 + F'_1 > 0$. We obtain the remaining two roots m_2^0 and m_3^0 for $F'_1 = -1$

$$m_2^0 = \frac{-(v_1 - F_1) - \sqrt{(v_1 + F_1)^2 + 4K^2p^2}}{2K} < 0,$$

$$m_3^0 = \frac{-(v_1 - F_1) + \sqrt{(v_1 + F_1)^2 + 4K^2p^2}}{2K} > 0.$$

The root m_3^0 is extraneous, and substituting m_2^0 into (11) we obtain $\Omega_2^0 < 0$. Hence, according to the above, both roots are stable or extraneous for any other initial parameters also.

Finally, the criterion of stability looks as follows: $F'_1 > -1$, or in different notation

$$f'_1 < \frac{A}{\varkappa(W_0 - W_{\text{H}})^2} = \frac{v_1}{W_0 - W_{\text{H}}}.$$

It can be seen from the expression obtained that when the velocity v_1 changes from small to large values, the unstable freezing mode changes to a stable mode. Consequently, the unstable mode e.g., freezing with the formation of ice layers) must be expected for small velocities. Similarly, a change in the initial moisture content from small values to high values leads, other conditions being equal, to a loss in the stability of the solution. Consequently, nonuniform freezing is more likely to be observed at high initial moisture contents. These properties of the stability criterion are in good qualitative agreement with the experimentally established facts (see, e.g., [9, 10]), characterizing the formation of a cryogenic texture: a) an increase in the rate of freezing is characterized by more uniform formation of ice, and b) an increase in the initial moisture content facilitates more intense formation of ice.

The absence in this criterion of the perturbation parameter p (6) enables us to conclude that the loss in stability of the freezing process may lead to the formation of layers of ice both parallel and perpendicular to the freezing front.

It should be emphasized that the stability criterion obtained only reflects the necessary conditions for ice streaks to form. In other words, the criterion describes the thermal situation in which the loss instability may lead to segregation ice separation. In particular, another case is possible when the loss instability leads to a periodic distribution of the ice content over the length of the specimen, without the formation of ice inclusions. To determine the sufficient conditions for the formation of streaks it is obviously necessary to carry out a physical-mechanical analysis of the processes which accompany the crystallization of moisture in the ground [11].

In conclusion we note that the development of a fairly reliable method for the experimental determination of the function f would enable the criterion obtained to be used in practical predictive calculations.

NOTATION

s(t), coordinate of the moving front; L, length of the specimen; k, moisture conductivity; W, moisture content; $\varkappa$, heat of phase transition; W_H, amount of unfrozen water; q, flow of moisture from the melted zone into the frozen zone; v, velocity of motion of the front; T_i, temperature; Q_i, heat flux; λ_i, thermal conductivity; a_i, thermal diffusivity (i = 1 is the frozen zone and i = 2 is the melted zone); α, the mass transfer coefficient; and T_H the initial temperature.

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